Condensation of Energy in Two Dimensional Turbulence

Turbulence is a state of spatio-temporal disorder characteristic of energetic fluid flows. Occuring in a wide variety of situations, it plays an essential role in various physical and industrial problems. Atmospheric science, aeronautics, oceanography and astrophysics provide obvious examples. Inviscid invariants play a key role in understanding the physics of turbulence. An inviscid invariant is a quantity conserved by the the Navier–Stokes (N–S) equations in the absence of forcing and damping. Kinetic energy, $E = \frac{1}{2}|\vec{u}|^2$, is an example, where $\vec{u}(\vec{x},t)$ is the fluid velocity. For three dimensional (3D) turbulence it is the principle invariant for most applications.

From a statistical physics perspective, turbulence should be thought of in terms of the transport of inviscid invariants between different scales. Invariants are typically injected and dissipated at very different scales. Viscosity dissipates efficiently at very small scales while external friction acts mainly at large scales. Nonlinear interactions transport inviscid invariants from the source scale to the dissipation scale, a process referred to as a cascade. In 3D turbulence, energy cascades from large scales to small to be dissipated by viscosity. A constantly forced flow reaches a stationary state with energy injection balanced by energy dissipation. There is no detailed balance. Rather there is a range of scales, known as an inertial range, between the forcing and dissipation scales through which an energy flux flows.

The two-dimensional (2D) N–S equations have additional invariants, foremost among which is the enstrophy, $H = |\nabla \times \vec{u}|^2$. They modify the physics considerably. Most importantly, in 2D flows, nonlinear interactions transfer energy primarily from the forcing scale to *larger* scales where it is ultimately dissipated by external friction while the enstrophy goes to small scales to be removed by viscosity. Energy flow to larger scales is called an inverse cascade while enstrophy flow to small scales is called a direct cascade. In 1967, Kraichnan made dimensional predictions for the energy spectrum, $E(k)$, in the stationary state where the fluxes of energy and enstrophy in the respective inertial ranges, denoted by ε and η, are constant. In the direct cascade, $E(k) \sim \eta^{\frac{2}{3}} k^{-3}$. In the inverse cascade, $E(k) \sim \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$. By transporting energy from incoherent small scales to larger scales, inverse cascades facilitate the formation of large scale coherent structures.

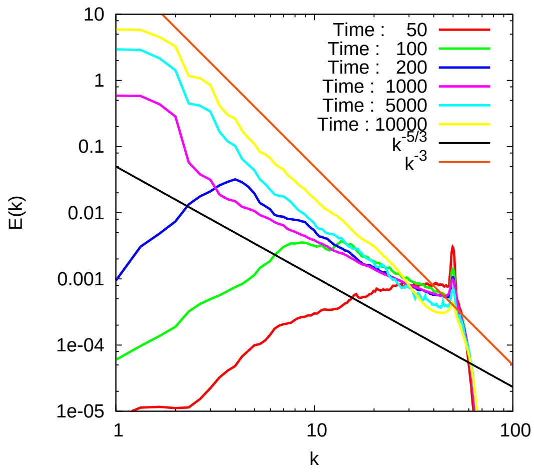

Figure 1: $E(k)$ as a function of time.

In most studies of the 2D inverse cascade, the size of the largest vortices is limited by external friction. As a vortex grows, the drag on the fluid layer increases until it eventually balances the energy flux carried by the inverse cascade and the vortex cannot grow further. If, however, the external drag is decreased sufficiently or removed entirely then vortices continue growing until they reach the system size. This is the box size in a simulation or the container size in the laboratory. In this situation, the inverse cascade cannot proceed further resulting in accumulation or “condensation” of energy at large scales. We performed a series of numerical experiments to investigate this effect. Such finite size effects provide a mechanism for strongly enhancing the stability and coherence of large scale structures in 2D turbulence. In fact turbulence can be entirely suppressed at the largest scales resulting in a large scale flow which is smooth and effectively deterministic.

Fig. 1 shows the energy spectrum, $E(k)$, as a function of time in the absence of external friction. At early times, it scales as $k^{-\frac{5}{3}}$ as expected from Kraichnan’s theory. When the inverse cascade reaches the system size, without large scale friction to provide

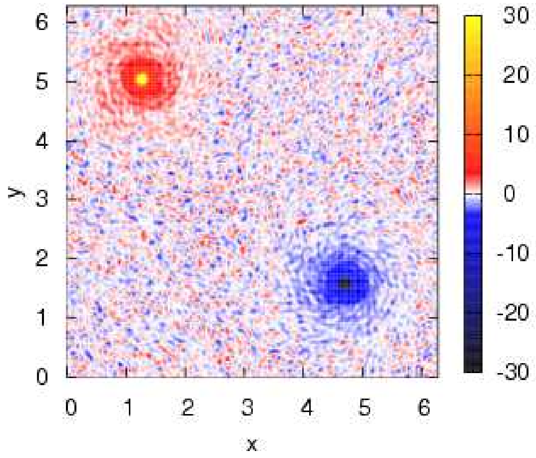

Figure 2: Vortex dipole resulting from energy condensation.

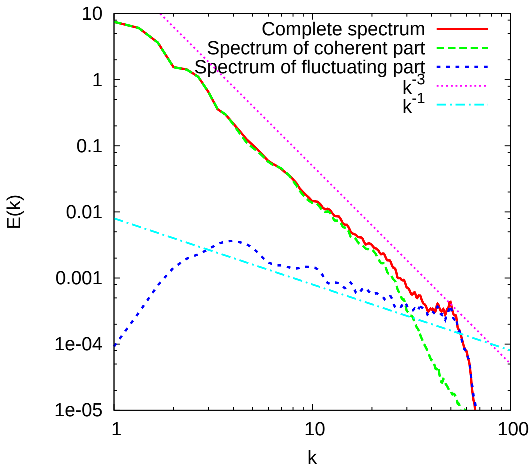

Figure 3: Spectra of the coherent and fluctuating components after condensation.

dissipation, energy accumulates at large scales. The spectrum slowly crosses over to k^{-3}, characteristic of smooth large scale flow. The late time vorticity field is shown in Fig. 2. Clearly the effect of the accumulation is to produce a very intense vortex dipole.

Once the large scale dipole emerges it is very stable. If the dissipation is entirely absent, the amplitude of the vortices continues to grow indefinitely at a rate proportional to $\sqrt{t}$. This is expected given that the total enstrophy of the system grows linearly in time (constant injection). If there is some small amount of dissipation present, then the growth eventually saturates. Small scale fluctuations produced by the forcing remain in the system. As the amplitude of the condensate grows, however, it comes to completely dominate the fluctuating part so that at large scales, the flow no longer appears turbulent. In this regime, one can easily use a wavelet transform to split the vorticity into coherent and fluctuating components. Once finds that the coherent dipole accounts for up to 99% of the total energy of the flow at late times. It is well known from studies of decaying turbulence that coherent vortices tend to produce steeper spectra. This is true in this case also. The k^{-3} spectrum which replaces the Kraichnan spectrum after condensation occurs is entirely due to the vortex dipole. See Fig. 3 which compares the spectra of the total, coherent and fluctuating flows. Interestingly, the fluctuations have a k^{-1} spectrum which suggests that, in the presence of a strongly developed condensate, the small scale vorticity tends to be passively advected by the large scale flow. Furthermore, as the coherent flow strengthens, the r.m.s. background vorticity fluctuation decreases in absolute terms. Thus the large scale coherent structures really suppress small scale turbulence.

The spatial structure of the vortices is such that the vorticity seems to decrease as a power law with distance from the vortex centre. The exponent is approximately 1.25, a number which currently lacks a convincing theoretical explanation. This vorticity profile is robust, growing self-similarly in time, with very little fluctuation. We are presently exploring how these results may be relevant to quasi-2D systems of more practical relevant such as the quasi-geostrophic equation in atmospheric science or the Hasegawa-Mima equation in plasma physics which are structurally and phenomenological similar to 2D hydrodynamics.

References

[1] M. Chertkov, C Connaughton, I. Kolokolov, and V. Lebedev. Growing condensate in two-dimensional turbulence. Submitted Phys. Rev. Lett., 2007, nlin/0612052.

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